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Game theory is a mature field of applied mathematics. It formalizes the conflict between competing agents, and has found applications ranging from economics through to biology [1,2]. Quantum information is a young field of physics. Regarding information as a physical quantity, rather than mathematical entity, has lead to concepts such as quantum computation [3]. Recently the first efforts have been made to combine these fields; the fusion may lead to new insights into the nature of information [4–6]. For two-player games, it has been found that when the allowed 'moves' are extended to include everything quantum mechanically possible, then the predominant strategies in the game can disappear, and only reappear if the players degrade the quantum *coherence*. Here we present the first study of quantum games with more than two players. We demonstrate that such games can exhibit 'coherent' equilibrium strategies which have no analogue in classical games, or even in two-player quantum games. These equilibria are generally of a cooperative nature: quantum players can exploit their environment highly efficiently through the use of collaborative strategies.

It has been known for some time that various quantum processes can be usefully thought of as games. Quantum cryptography, for example, is readily cast as a game between the individuals who wish to communicate, and those who wish to eavesdrop [7]. Quantum cloning has been thought of as a physicist playing a game against nature [8], and indeed even the measurement process itself may be thought of in these terms [9]. Furthermore, Meyer [10,11] has pointed out that quantum algorithms may be usefully thought of as games between classical and quantum agents. Against this background, it is natural to seek a unified theory of games and quantum mechanics. Such a theory might lend insight into biological and chemical processes occurring in the quantum regime; it would certainly provide a fuller understanding of the physics of information [3].

The fundamental unit of classical information is the *bit*. The corresponding unit of quantum information is the 'qubit' − a general quantum superposition of '0' and '1', $\alpha_0|0\rangle + \alpha_1|1\rangle$. In multi-qubit systems, superposition gives rise to *entanglement*: qubits are entangled if their states cannot be defined independently from one another. Whereas a pair of classical bits must be in one of the four states $\{00, 01, 10, 11\}$, a pair of qubits can be in a state, such as $\frac{1}{\sqrt{2}}(|0\rangle\otimes|0\rangle+|1\rangle\otimes|1\rangle)$, which cannot be factorized into two separate qubit states. The interdependence remains even when the two qubits are far apart - this is the has been identified as a crucial resource in quantum communication, quantum computation and error-correction, and some forms of quantum cryptography [3]. Here we will see that when the resources controlled by competing agents are entangled, they can cooperate to perfectly exploit their environment (i.e. the 'game').

Formally a *game* involves of a number of agents or *players*, who are allowed a certain set of moves or *actions*. The *payoff function* \$() specifies how the players will be rewarded after they have performed their actions. The i^{th} player's *strategy*, s_i, is her procedure for deciding which action to play, depending her information. A *strategy space*, $S = \{s_i\}$, is the set of strategies available to her. A *strategy profile* $s = (s_1, s_2, .., s_N)$ is an assignment of one strategy to each player. An *equilibrium* is a strategy profile with a degree of stability: e.g. in a Nash equilibrium no player can improve her expected payoff by unilaterally changing her strategy. The study of equilibria is fundamental in understanding a game. The games we consider here are *static*: they are played only once so that there is no history for the players to consider. Moreover, each player has complete knowledge of the game's structure. Thus the set of allowed actions corresponds directly to the space of deterministic strategies.

Our procedure for quantizing games is a generalization of the elegant scheme introduced by Eisert *et al.* [12,13]. We reason as follows. Game theory, being a branch of applied mathematics, defines games without reference to the physical universe. However, quantum mechanics is a physical theory, and must be applied to a physical system. We therefore begin by creating a physical model for the games of interest. A very natural way to do this is by considering the flow of information, see Fig 1(a).

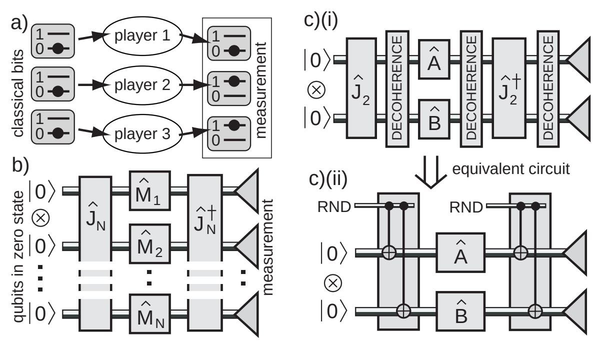

FIG. 1. a) a physical model for a game in which each player has two possible actions: we send each player a classical 2-state system (a bit) in the zero state. They locally manipulate their bit in whatever way they wish: under classical physics their choices are really just to flip, or not to flip. They then return the bits for measurement, from which the payoff is determined. b) Our N-player quantized game. Throughout this paper, 'measurement' means measurement in the computational basis, $\{|0\rangle, |1\rangle\}$. c) The effect of introducing total *decoherence* of the quantum information. RND denotes a random classical bit, the vertical lines denote CONTROL-NOT.

This classical physical model is then to be quantized. Our quantization procedure is the most natural one that meets the following requirements: (a) The classical information carriers (bits) are to be generalized to quantum systems (qubits). (b) These qubits are to be mutually entangled [14]. (c) The resulting game must be a *generalization* of the classical game: the identity operator $\hat{I}$ should *correspond* to 'don't flip', and the bit-flipping operator $\hat{F} = \hat{\sigma}_x$ should *correspond* to 'flip' [15], in the sense that when all the players restrict themselves to choosing from $\{\hat{F}, \hat{I}\}$, then the payoffs of the classical game are recovered. To simultaneously meet requirements (b) and (c), we employ a pair of entangling gates as shown in Fig 1(b), and insist that $\hat{J}$ commutes with any operator formed from $\hat{F}$ and $\hat{I}$ acting in the subspaces of different qubits. If we restrict ourselves to unitary, maximally entangling gates [16] that act symmetrically on ones and zeros, then we may specify $\hat{J}$ without loss of generality [17]: $\hat{J} = \frac{1}{\sqrt{2}}(\hat{I}^{\otimes N} + i\hat{F}^{\otimes N})$.

In comparing the quantum and classical games, the choice of strategy space is fundamental. The classical game is to be embedded in the quantum game, therefore the space should include playing the 'classical' actions $\{\hat{I},\hat{F}\}$, but in principle we could choose any superset of this 'classical' space. For example, we could study the consequences of giving some players a larger strategy space than others [11,13]. Alternatively, we could force new winning strategies to emerge by permitting the players some strategy whilst forbidding the logical counter strategy [13,18]. However, the approach we take here is to allow all of the players to perform any action on their qubits which is quantum mechanically possible, including adjoining arbitrarily large ancillas, performing measurements and applying operations conditioned on the outcomes of those measurements. This a very natural generalization of our classical model, where the only restrictions on the actions of the players were those imposed by classical physics. General quantum operations are represented by trace-preserving, completely-positive maps, and we denote the space of strategies corresponding to all such operations by S_{TCP}.

In traditional game theory, there is a fundamental distinction between so-called 'pure' strategies, in which players choose their actions deterministically, and 'mixed' strategies which can involve probabilistic choices. An important consequence of adopting a general quantum model is that the players can implement any probabilistic strategy entirely deterministically through the use of ancillary qubits. For example, such qubits could function as a random number generator controlling the operations applied to the primary qubit. Thus, all quantum strategies could be argued to be 'deterministic'. Even so, there is a subset of S_{TCP} that is in many ways analogous to the classical deterministic strategies, namely the set of all strategies which correspond directly to a unitary action. Strategies from this subset, which we label S_U, imply coherent manipulations of the local qubits, i.e. manipulations without the addition of ancillary qubits. Another way of identifying the set S_U is that they are precisely the strategies that do not destroy any of the entanglement introduced by the $\hat{J}$ gate [19].

For the multi-player games we consider below, we find equilibria for all of S_{TCP} consisting of strategies drawn only from S_U. We will refer to these special equilibria as *pure*, or *coherent*. They are *fundamentally* quantum mechanical, in that they disappear when the quantum correlations implicit in the entangled states are replaced with classical correlations, as in Fig. 1(c). In analogous two-player games (where both players are permitted S_{TCP}) it is impossible for 'pure' equilibria [18] to occur – instead equilibria exist only when the players choose to degrade the entanglement. Unsurprisingly therefore, those equilibria do persist in the Fig 1(c) variant.

Consider the classical N-player Minority Game [20]. Here each player privately chooses between two options, say '0' and '1'. The choices are then compared and the player(s) who have made the minority decision are rewarded (by one point, say). If there is an even split, or if all players have made the same choice, then there is no reward. The structure of this game reflects many common social dilemmas, for example choosing a route in rush hour, choosing which evening to visit an overcrowded bar, or trading in a financial market.

Let us focus on the 4-player Minority Game. Classically, the players have no better strategy than to choose randomly between the '0' and '1' actions. The expected payoff for each player is then one eighth of a point, i.e. the game only 'pays out' half the time. However, the quantum game exhibits a fundamentally new kind of equilibrium, one in which each player has expected payoff 0.25: twice the performance of the classical game and the logical maximum for a cooperative solution. One example [21] of such an equilibrium is the strategy profile $s = (\frac{1}{\sqrt{2}}(\hat{I} + i\hat{\sigma}_x), \frac{1}{\sqrt{2}}(\hat{I} + i\hat{\sigma}_x), \frac{1}{\sqrt{2}}(\hat{I} + i\hat{\sigma}_x), \frac{1}{\sqrt{2}}(\hat{I} - i\hat{\sigma}_y))$. With these choices, the final state prior to measurement is $\frac{i}{2}(|1000\rangle + |0100\rangle + |0010\rangle - |1110\rangle)$, i.e. an equal superposition of four states, one optimal to each player. The reasoning below shows that, even with the most general strategy space S_{TCP}, this s is a Nash equilibrium: no player can improve her expected payoff by unilaterally changing her choice of strategy.

1. Note that the Minority Game has the special property that the same expected payoffs result whether or not we apply the second gate, $\hat{J}^\dagger$, prior to measurement. This can be seen by noting that $J^\dagger$ transforms any basis vector $|abcd\rangle$ only within the sub-space spanned by $\{|abcd\rangle, |\bar{a}\bar{b}\bar{c}\bar{d}\rangle\}$, where $\bar{x} = NOT(x)$. Since both $|abcd\rangle$ and $|\bar{a}\bar{b}\bar{c}\bar{d}\rangle$ have the same payoff value, the expected payoff is left invariant by the $\hat{J}^\dagger$.

2. Because of (1), we can focus attention on the state prior to $J^\dagger$. This state has the property that measurement of any three of the four qubits will yield

one of the eight outcomes, (000), (001), ...,(111), with *equal* probability. This must remain true regardless of what local action was performed on the fourth qubit. Violation of this physical principle would mean that entanglement could be used for faster than light information transfer, for example.

3. Six of these eight outcomes are *unwinnable* by the fourth player: if, for example, measurement of the first three qubits yields (001), then neither a '0' or a '1' will put the fourth player in the minority. Thus, because of the equal weighting of the outcomes, her expected payoff cannot exceed $\frac{1}{4}$. *But this is just the payoff each player has with the originally proposed strategy profile.*

This equilibrium is optimal and fair: the game always pays out the maximal amount *and* the expected payoff for each of the players is the same. In the classical game, this *can* be achieved, but only by invoking additional shared resources [22]. It is interesting to ask, are there games with coherent quantum equilibria whose performance cannot be matched *even* by invoking arbitrary additional classical resources? To answer this question, we construct a game having a dominant-strategy equilibrium.

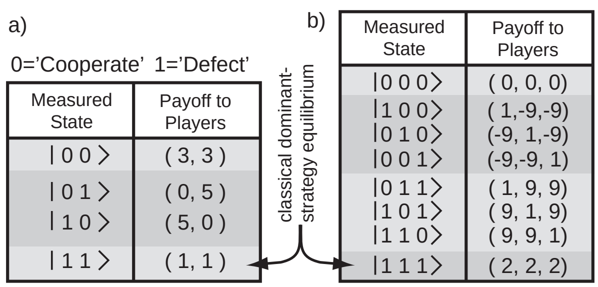

FIG. 2. Games possessing a dominant-strategy equilibrium: a) table defining the payoffs in Prisoner's Dilemma. Either player reasons thus: 'If my partner were to cooperate, my best action would be to defect. If he were to defect, my best action is still to defect. Thus I have a dominant strategy: "always defect." ' b) Table defining the payoffs for a three-player game. Classically, each player has the dominant strategy 'choose 1'. Consequently, each player's payoff is just 2 points (despite the existence of strategy profiles, such as 'choose 1 with probability 80%', for which all the players have greater expected payoffs). However in the quantum game, radically superior new coherent equilibria arise.

A player has a *dominant* strategy if this strategy yields a higher payoff than any alternative, *regardless* of the strategies adopted by other players. A rational player will always adopt such a strategy, regardless of any additional information: even prior conversation with other players makes no difference (unless we introduce some kind of binding contract, which amounts to switching to another game). Most games, including the Minority Game considered earlier, do not possess dominant strategies. If every player has a dominant strategy, then the game's inevitable outcome is the *dominant-strategy equilibrium.* The famous Prisoner's Dilemma, shown in Fig 2(a), has the dominant-strategy equilibrium ('defect','defect'). As noted above, no maximally entangled two-player quantum game can have equilibria when the strategy space is S_U. Thus, quantization of Prisoner's Dilemma removes the dominant-strategy equilibrium [12], but does not provide alternative coherent equilibria that might offer better payoffs to the players.

To investigate the multi-player case, we quantize the game of Figure 2(b). We find that coherent equilibria *do* occur. The classically inevitable outcome, now written as $(\hat{F}, \hat{F}, \hat{F})$, remains as a Nash equilibrium – but other, radically superior equilibria emerge. For example, the profile $s = (\hat{I}, \frac{1}{\sqrt{2}}(\hat{\sigma}_x + \hat{\sigma}_z), \hat{\sigma}_x)$, with expected payoffs $(5, 9, 5)$, is a Nash equilibrium (and is *strict* for players A and C: any unilateral deviation necessarily leads to *reduction* in their expected payoffs). Thus, as in the Minority Game, we find that equilibria consisting of S_U actions exist which drastically out-perform any classical strategies. The proof runs as follows.

Let $|\psi\rangle = (\hat{I} \otimes \frac{i}{\sqrt{2}}(\sigma_x + \sigma_z) \otimes \hat{I})\hat{J}|000\rangle$ be the state after the actions of players A and B, and suppose that player C applies a general open quantum operation $\mathcal{R}$, i.e. a completely positive, trace-preserving map on density operators. By the Kraus representation theorem [23], we can write $\mathcal{R}(\rho) = \sum_k A_k \rho A_k^\dagger$, under the restriction $\sum_k A_k^\dagger A_k = I$. We may think of this expansion as representing a k-outcome measurement, where it is allowed to perform unitary operations conditioned on the outcome of the measurement. The state-change corresponding to outcome k is given by $|\psi\rangle \mapsto (\langle\psi|A_k^\dagger A_k|\psi\rangle)^{-1} A_k|\psi\rangle$. Since player C only applies local operations, the most general $A_k = I \otimes I \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. But it is then simple to show, by applying this A_k followed by the gate $J^\dagger$, that player C's expected payoff is maximized only if $A_k \propto \sigma_x$. Therefore, the only strategy for player C which maximizes her expected payoff for every one of her measurement outcomes is, up to global phase, σ_x. By repeating similar arguments for players A and B, we verify that s is indeed a Nash equilibrium for the full quantum strategy space S_{TCP}.

To conclude, we have performed the first investigation of multi-player quantum games. We have determined that such games can exhibit forms of 'pure' quantum equilibrium which have no analogue in classical games, or even in two-player quantum games. We have explicitly examined the Minority game, and a game analogous to to Prisoner's Dilemma. Our observation of purely coherent equilibria paves the way for an investigation of iterated quantum games.

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[1] For an introduction see, e.g., Eric Rasmusen, *Games and Information*, (Blackwell, 1995).
[2] Nowak, M. A. & Sigmund, K. Phage-lift for game theory. *Nature* **398**, 367 (1999).
[3] For a review see, e.g. Bennett, C. H. & DiVincenzo, D. P. *Nature* **404**, 247 - 255 (2000).
[4] Ball, P. Everyone wins in quantum games. *Nature*, Science Update, 18 Oct. 1999.
[5] Peterson, I. Quantum Games, Taking Advantage of Quantum Effects to Attain a Winning Edge. *Science News* **156**, 334 (1999).
[6] Collins, G. Schrödinger's Games. *Sci. Am.*, Jan. 2000.
[7] Ekert, A. K. *Phys. Rev. Lett.* **67**, 661 (1991).
[8] Werner, R. F. *Phys. Rev. A* **58**, 1827 (1998).
[9] Deutsch, D., Quantum theory of probability and decisions. *Proc. R. Soc. Lond. A* **455**, 3129 (1999).
[10] Meyer, D. A. Quantum games and quantum algorithms. Preprint at http://xxx.lanl.gov/abs/quant-ph/0004092.
[11] Meyer, D.A. Quantum strategies. *Phys. Rev. Lett.* **82**, 1052 (1999).
[12] Eisert J. & Wilkens, M. Quantum Games. Preprint at http://xxx.lanl.gov/abs/quant-ph/0004076.
[13] Eisert, J., Wilkens, M. & Lewenstein, M. Quantum games and quantum strategies. *Phys. Rev. Lett.* **83**, 3077 (1999).
[14] Without entanglement, the quantized form of Fig 1(a) remains equivalent the classical probabilistic game: the only effect of a quantum action on a qubit is to alter the probability of measuring '1' on that qubit *alone*.
[15] The most general quantum bit-flip operator (up to a global phase) is $\cos(\theta)\hat{\sigma}_x + \sin(\theta)\hat{\sigma}_y$. Our $\hat{F} = \sigma_x$, but any other choice is equally valid, and would simply result in a trivial rotation of the features which we discover.
[16] Since we are interested in purely multipartite entanglement, we call a (pure) state maximally entangled if it is equivalent via local unitary operations to the GHZ-type state, $1/\sqrt{2}(|00\cdots 0\rangle + |11\cdots 1\rangle)$.
[17] Any other $\hat{J}$ meeting these conditions would be equivalent, via local unitary operations, to our $\hat{J}$. Therefore the resulting game equilibria would be equivalent, via the same operators, to the equilibria we discover.
[18] Benjamin, S. C. & Hayden, P. Comment on "Quantum games and quantum strategies". Preprint at http://xxx.lanl.gov/abs/quant-ph/0003036.
[19] If the action is not drawn from S_U then the resulting state is necessarily not equivalent via local unitary operations to a GHZ state. Such a state is therefore no longer 'maximally entangled' in our sense.
[20] A survey of recent work on the classical game, which is usually studied in its iterative form, is available from http://www.unifr.ch/econophysics/minority/.
[21] Interestingly, the complexity of the four player game is such that many Nash equilibria occur. Whilst players may use criteria, such as maximum projection into the subspace of 'classical' moves $\{\hat{I}, \hat{F}\}$, to establish a focal point, this becomes a psychological question.
[22] If we generate a pair of random classical bits, and send duplicates of these bits to each player, then Nash equilibria exist wherein the players use these bits to decide which player should be in the minority.
[23] K. Kraus. *States, Effects, and Operations: Fundamental Notions of Quantum Theory.* Lecture Notes in Physics, Vol. 190. (Springer-Verlag, Berlin, 1983).